3 Sobolev Spaces

Exercise 3.1 (Lemma of Gagliardo)**.** Let $f_1, \dots, f_{k+1} \in L^k(\mathbb{R}^k)$ and for every $x \in \mathbb{R}^{k+1}$ and $1 \leq i \leq k+1$, define

$$\widehat{x_i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{k+1}) \in \mathbb{R}^k.$$

Prove that the function $f : \mathbb{R}^{k+1} \to \mathbb{R}$ defined pointwise as

$$f(x) = \prod_{i=1}^{k+1} f_i(\widehat{x_i})$$

is in $L^1(\mathbb{R}^{k+1})$ and $\|f\|_{\mathrm{L}^1(\mathbb{R}^{k+1})} \leq \prod_{i=1}^{d} \|f_i\|_{\mathrm{L}^k(\mathbb{R}^k)}$.

Exercise 3.2. Let $\Omega \subset \mathbb{R}^d$ be open. Prove the following statements :

— Let $1 \leq p < q \leq \infty$ and $u \in L^{q_1}(\Omega) \cap L^{q_2}(\Omega)$, then for any $\alpha \in [0,1]$ and p such that $\dfrac{1}{r} = \dfrac{\alpha}{p} + \dfrac{1-\alpha}{q}$ we have

$$\|u\|_{L^r(\Omega)} \leq \|u\|_{L^p(\Omega)}^{\alpha} \|u\|_{L^q(\Omega)}^{1-\alpha}.$$

In particular $u \in L^r(\Omega)$ for all $p \leq r \leq q$.

— Let $1 \leq \alpha < \beta \leq 1$ and $u \in C^{0,\alpha}(\Omega) \cap C^{0,\beta}(\Omega)$, then for any $t \in [0,1]$ and γ such that $\gamma = t\,\alpha + (1-t)\beta$ we have

$$[u]_{C^{0,\gamma}(\Omega)} \leq [u]_{C^{0,\alpha}(\Omega)}^{t} [u]_{C^{0,\beta}(\Omega)}^{1-t},$$

so $u \in C^{0,\gamma}(\Omega)$ for any $\alpha \leq \gamma \leq \beta$.

Exercise 3.3. Let $n \geq 2$. Let $\Omega = B(0, 1/e)$ be the ball of radius $1/e$ of $\mathbb{R}^d$ centred at the origin. Verify that the function

$$u(x) = \log\left(\log\left(\frac{1}{|x|}\right)\right)$$

is in $W^{1,d}(\Omega)$ and $L^q(\Omega)$ for any $1 \leq q < \infty$ but not in $L^\infty(\Omega)$.

Remark. This shows that the inclusion $W^{1,n}(\Omega) \subseteq L^\infty(\Omega)$ is false in general.

Exercise 3.4. Show that the inclusion $W^{1,p}(\Omega) \subset L^q(\Omega)$ for $1 \leq q < p$ is, in general, false whenever Ω is unbounded.

Exercise 3.5. Let $p > d$. Show that if $f,\, g \in W^{1,p}(\mathbb{R}^d)$ then $fg \in W^{1,p}(\mathbb{R}^d)$.